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Triangle Inscribed-Triangle Picking

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A tale of three points

In 1865, James Joseph Sylvester proved [8] that the average area of a random triangle, whose vertices are picked inside a given triangle of unit area, is equal to 1/12. This problem, originally proposed by S. Watson, and known as Triangle Triangle Picking, is one of the earliest examples of geometric probability [9]. Many similar problems have been proposed [4, 13], including Sylvester's own four-point problem [14], which asks for the probability that four random points in a convex shape have a convex hull that is a quadrilateral. Problems involving properties of inscribed geometric figures have also been studied; for example, questions related to the average distance of inscribed points appear in [2], while in [6] the average area and perimeter of a triangle inscribed in a circle is found.

Here we consider a class of such problems where the interior polygon has its vertices on the edges of the base convex polygon, with one vertex per side. In particular we look at the properties of a random triangle that is inscribed in a fixed triangle. The probability distribution function and the moments of the area of the inscribed triangle are derived in this paper.

An application of barycentric coordinates

A simple and effective way of describing triangles within triangles is to use barycentric coordinates. Suppose the vertices of a triangle are denoted by the vectors $\vec{A}, \vec{B}, \vec{C}$. The barycentric coordinates [10] of a point \vec{P} , with respect to the triangle *ABC*, are (α, β, γ) if $\vec{P} = \alpha \vec{A} + \beta \vec{B} + \gamma \vec{C}$, and $\alpha + \beta + \gamma = 1$. Bottema's theorem [3] gives the area of a triangle if the barycentric coordinates of its vertices are specified with respect to another triangle of known area.

Theorem 1 (Bottema). Let $|\Delta ABC|$ represent the area of triangle ABC. Assume the vertices P_i of a triangle $P_1P_2P_3$ have barycentric coordinates (x_i, y_i, z_i) , with respect to the triangle ABC, then

$$|\Delta P_1 P_2 P_3| = \left| \det \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \right| |\Delta ABC|.$$

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By using the above theorem we can easily calculate the moments of the area of the inscribed triangle.

Theorem 2. Given a unit triangle ABC, if three points R, S, and T are chosen uniformly and independently on the sides AB, BC, and CA respectively then the average area of RST is 1/4.

Proof. Consider an inscribed triangle whose vertices R, S, T, are defined as

$$\begin{cases} \vec{R} = \vec{B} + r \ \vec{BC} \\ \vec{S} = \vec{C} + s \ \vec{CA} \\ \vec{T} = \vec{A} + t \ \vec{AB} \end{cases}$$

where r, s, t are uniformly and independently distributed random numbers in [0, 1].



Figure 1. Triangle ABC, and an inscribed triangle RST.

In this case, the points R, S, T are respectively given by barycentric coordinates (0, r, 1 - r), (1 - s, 0, s) and (t, 1 - t, 0). Now we define Q(r, s, t) as the area of *RST*. Therefore, by Bottema's theorem

$$Q(r, s, t) = \det \begin{bmatrix} 0 & r & 1-r \\ 1-s & 0 & s \\ t & 1-t & 0 \end{bmatrix} = rst + (1-r)(1-s)(1-t).$$
(1)

Now we will set out to calculate E[Q], the expected value of Q(r, s, t). Since r, s, t are independent, the expected value of rst, and (1 - r)(1 - s)(1 - t) can be represented by the product of the expected values of r, s, t. Specifically, the first moment of Q evaluates to

$$E[Q(r, s, t)] = E[rst + (1 - r)(1 - s)(1 - t)]$$

= $\int_0^1 \int_0^1 \int_0^1 (rst + (1 - r)(1 - s)(1 - t)) dr ds dt$
= $\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \frac{1}{4}.$

As a result, the first moment of the area of the inscribed triangle is $E[|\Delta RST|] = 1/4$.

The *n*th moment, $E[Q^n(r, s, t)]$. To derive the *n*th moment of the area, we expand $Q^n(r, s, t)$ using the binomial theorem,

$$Q^{n}(r, s, t) = [rst + (1 - r)(1 - s)(1 - t)]^{n}$$

= $\sum_{k=0}^{n} \frac{n!}{k!(n - k)!} (rst)^{n-k} ((1 - r)(1 - s)(1 - t))^{k}$
= $\sum_{k=0}^{n} \frac{n!}{k!(n - k)!} r^{n-k} (1 - r)^{k} s^{n-k} (1 - s)^{k} t^{n-k} (1 - t)^{k}.$

The average value of $(rst)^{n-k}((1-r)(1-s)(1-t))^k$ is found using the Euler beta function [11]: $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = (x-1)!(y-1)!/(x+y-1)!$. By letting x = n - k + 1, y = k + 1, we get

$$E[r^{n-k}(1-r)^k] = \int_0^1 r^{n-k}(1-r)^k dr = \frac{(n-k)!k!}{(n+1)!}.$$

Thus, μ'_n , the *n*th raw moment of (1), can now be expressed as

$$\mu'_{n} = E[Q^{n}(r, s, t)] = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{(n-k)!k!}{(n+1)!}\right)^{3}$$

$$= \frac{a(n)}{(n+1)(n+1)!^{2}},$$
(2)

where

$$a(n) = \sum_{k=0}^{n} (n-k)!^{2}k!^{2},$$
(3)

is the Sloane integer sequence A279055 [7]. Therefore, $E[|\Delta RST|^n] = \mu'_n$. In particular, the first few values of (2) are as shown in Table 1.

n	1	2	3	4	5	6	7
μ'_n	$\frac{2}{2\cdot 2!^2}$	$\frac{9}{3\cdot 3!^2}$	$\frac{80}{4\cdot 4!^2}$	$\frac{1240}{5\cdot 5!^2}$	$\frac{30240}{6 \cdot 6!^2}$	$\frac{1071504}{7\cdot7!^2}$	$\frac{51996672}{8 \cdot 8!^2}$

Table 1. The first seven raw moments of Q(r, s, t).

A Monte Carlo simulation of the probability density function

A Monte Carlo simulation [1, 5] was conducted to numerically study and validate the theoretical findings for the distribution of the area of a randomly generated inscribed triangle. The output of the simulation is the experimental probability density function, with a sharp peak at 1/4, as depicted in Figure 2. We derive the elementary functions that produce this curve (nicknamed "The Witch's Sorting Hat," or "The Shark Fin") in the next section.



Figure 2. Experimental probability density function of Q.

Piecewise-defined probability distributions are commonly seen in geometric probability. The most notable example is in Sylvester's Triangle Triangle Picking, whose PDF is also split at 1/4. The significance of this particular value is explained in the following section. A similar case of extreme slope discontinuity is also seen in Cube Line Picking [12].

To test the simulation itself we ran an experiment for the mean area. The observed average value of $Q(r, s, t) = |\Delta RST|$ is expected to approach $\mu'_1 = 1/4$ as the sample size is increased. A Java application was employed to study the deviation of the experimental average from its theoretical value, err = $E_{\exp}[Q(r, s, t)] - E[Q(r, s, t)]$. From central limit theorem err has an approximately normal distribution with a standard deviation of σ/\sqrt{n} , where $\sigma = (\mu'_2 - {\mu'_1}^2)^{1/2} = \sqrt{3}/12$, and *n* is the sample size. As such, $E_{\exp}[|\text{err}|]$, the experimental average value of |err|, is to approach $\sqrt{2/n\pi\sigma}$. We ran the simulation with sample of sizes of $n = 10^2$ to 10^8 and averaged |err| over 50 trials. For 10^8 samples, we observed $E_{\exp}[|\text{err}|] \approx 1.2 * 10^{-5}$, while $\sqrt{2/n\pi\sigma} \approx 1.1 * 10^{-5}$.

Cumulative and probability density functions

In this section, we will derive the cumulative density function, CDF, and the probability density function, PDF, of the area Q(r, s, t).

We have $\text{CDF}(c) = \text{Vol}\{(r, s, t) \in [0, 1]^3 | Q(r, s, t) \le c\}$. The surface defined by Q(r, s, t) = c is quadratic. To eliminate the cross terms and determine the surface type, we rotate the axes so that one axis is along the rotation vector [1, 1, 1]. The other two axes can be, for example, in the direction of [1, -1, 0], and [1, 1, -2]. After the normalization of vectors the rotation formula becomes

$$\begin{pmatrix} r\\s\\t \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3}\\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3}\\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \tilde{r}\\\tilde{s}\\\tilde{t} \end{pmatrix},$$

and the surface Q = c is now given by $-\tilde{r}^2/2 - \tilde{s}^2/2 + (\tilde{t} - \sqrt{3}/2)^2 = c - 1/4$. Hence, for $c \in [0, 1/4)$ the surface is a hyperboloid of one sheet, for c = 1/4 it is a double cone, and for $c \in (1/4, 1]$ it is a hyperboloid of two sheets. For c < 1/4, CDF(c) is equal to the volume of a region similar to Figure 3a. For c > 1/4 it is equal to the volume of a region similar to Figure 3b.



Figure 3. Contour plots of Q = c for c = 1/5, 1/3 respectively. The family of hyperboloids are centered at (1/2, 1/2, 1/2), with axis of rotation t = s = r. The portion within the unit cube $[0, 1]^3$ is displayed.

To detail the integration steps we note that the equation for the surface Q(r, s, t) = c can be written in three ways:

$$rst + (1 - r)(1 - s)(1 - t) = c$$
(4)

$$(r+t-1)(s+t-1) = (t-1/2)^2 + c - 1/4$$
(5)

$$r = r(s, t, c) = \frac{c - st + s + t - 1}{s + t - 1}.$$
(6)

From the first equation we see that the surface is symmetric with respect to the center of the unit cube, (1/2, 1/2, 1/2). This will be used to carry the integration over a half region and double the result. From the second equation we see that the shape of intersection of the surface with a plane $t = t_0$ is a right hyperbola, on an (r, s) plane and centered at $(1 - t_0, 1 - t_0)$, which divides the plane into three regions. The sign of the right hand side of (5), $K(t_0, c) = (t_0 - 1/2)^2 + c - 1/4$, will then be used to determine the regions that are included in the integration. If K > 0 then the solution of Q < c corresponds to the connected region between the two branches of the hyperbola, however, if K < 0, the solution corresponds to the complement of this region. At K =0 the hyperbola collapses to a pair of perpendicular lines. The third equation defines ras a function of s and t for the integration process. The Mathematica commands that we used for graphing and integration appear in the Appendix.

Derivation of CDF(*c*) and PDF(*c*) for $c \in (1/4, 1]$ In this case the region of integration is bounded by a hyperboloid of two sheets as in Figure 3b. A slicing plane $t = t_0$ cuts this surface at a hyperbola, and, following the discussion under (5), the region of integration is always between the two branches of the hyperbola

as $K(t_0, c) = (t_0 - 1/2)^2 + (c - 1/4) > 0$. The volume to calculate is that of the unit cube with two "scoops" removed from two corners, (0, 0, 0) and (1, 1, 1). We calculate the volume of the scoop near (1, 1, 1), double the result, and subtract it from the volume of the unit cube to get the required integral.

Note that the top face of the cube, where r = 1, cuts the surface (4) at st = c and as a result the integration limits at a fixed t will be from s = c/t to s = 1, and t will have a range from c to 1. Therefore the volume of the region in Figure 3a can be calculated as

$$CDF(c) = 1 - 2\int_{c}^{1}\int_{\frac{c}{t}}^{1} (1 - r(s, t, c))dsdt,$$
(7)

where r(s, t, c) is given by (6). We used Mathematica to perform this integration and arrived at the following expression

$$CDF(c) = c - (3c - \frac{1}{2}) \ln c - \frac{(4c - 1)^{\frac{3}{2}}}{3} \times \left(\tan^{-1} \left(\frac{1}{\sqrt{4c - 1}} \right) - \tan^{-1} \left(\frac{2c - 1}{\sqrt{4c - 1}} \right) \right).$$
(8)

To simplify (8) further we employ a Machin-like identity, namely, for $c > \frac{1}{4}$ we have

$$\tan^{-1}\left(\frac{1}{\sqrt{4c-1}}\right) - \tan^{-1}\left(\frac{2c-1}{\sqrt{4c-1}}\right) = \pi - 3\tan^{-1}\sqrt{4c-1}.$$

To see this note that derivative of both sides is $-(3/2)c^{-1}(4c-1)^{-1/2}$, and for c = 1 both sides are equal to zero. As a result, (8) simplifies to

$$CDF(c) = c - (3c - \frac{1}{2})\ln c + (4c - 1)^{\frac{3}{2}} \left(\tan^{-1}(\sqrt{4c - 1}) - \frac{\pi}{3} \right).$$

By differentiation we can find PDF(c)

$$PDF(c) = \frac{d}{dc}CDF(c) = 2\sqrt{4c-1}\left(3\tan^{-1}\sqrt{4c-1} - \pi\right) - 3\ln c.$$

Derivation of CDF(*c*) **and PDF**(*c*) **for** $c \in [0, 1/4)$ Due to the presence of a hole in the middle of the corresponding volume of integration calculating CDF(*c*) for 0 < c < 1/4 is more involved than the previous case. This volume can be broken into five parts according to how a plane $t = t_0$ intersects the hyperboloid. Summarizing the result we get

$$\frac{\text{CDF}(c)}{2} = \int_{0}^{c} \left(1 - \int_{0}^{1 - \frac{c}{1 - t}} r \, ds \right) dt + \int_{c}^{\frac{1 - \sqrt{1 - 4c}}{2}} \left(1 - \int_{0}^{1 - \frac{c}{1 - t}} r \, ds - \int_{\frac{c}{7}}^{1} (1 - r) \, ds \right) dt + \qquad (9)$$
$$\int_{\frac{1 - \sqrt{1 - 4c}}{2}}^{\frac{1 + \sqrt{1 - 4c}}{2}} \left(\int_{1 - \frac{c}{1 - t}}^{1} r \, ds \right) dt.$$

The first integral represents the case when the intersection of $t = t_0$ with the portion of hyperboloid in the unit cube shows only one branch of a hyperbola. In the second integral, two branches of a hyperbola appear and we calculate the area between the two. At the edge of the internal hole the hyperbola turns into a pair of perpendicular lines and we begin to calculate the area of the two symmetric disjoint regions. At the end of the hole a region similar to the second integral appears and finally the last region is similar to the first integral.

The edge of the internal hole, at which the hyperbola turns into a pair of lines, occurs at K(t, c) = 0. This gives $t = (1 \pm \sqrt{1 - 4c})/2$ and $r = s = (1 \mp \sqrt{1 - 4c})/2$.

Alternatively, we may delegate the segmentation of the integral (9) to Mathematica and use its Boole command. The result can be simplified using FullSimplify and TrigToExp commands. Using $(1 \pm \sqrt{1-4c})^2 = 2(1-2c \pm \sqrt{1-4c})$, and $\tanh^{-1}(a) = \frac{1}{2}\ln(\frac{1+a}{1-a})$, we finally arrive at

$$\text{CDF}(c) = c - (3c - 1/2) \ln c + (1 - 4c)^{3/2} \tanh^{-1} \sqrt{1 - 4c}.$$

Upon differentiation of CDF(c) the PDF is found to be

$$PDF(c) = -3 \ln c - 6\sqrt{1 - 4c} \tanh^{-1} \sqrt{1 - 4c}.$$

To summarize the results of (7), and (9), we have

$$CDF(c) = \begin{cases} c - (3c - \frac{1}{2})\ln c + (1 - 4c)^{3/2} \tanh^{-1} \sqrt{1 - 4c}, & \text{for } 0 \le c \le \frac{1}{4} \\ \frac{1}{4}(1 + \ln 4), & \text{for } c = \frac{1}{4} \\ c - (3c - \frac{1}{2})\ln c + (4c - 1)^{3/2} (\tan^{-1} \sqrt{4c - 1} - \frac{\pi}{3}) & \text{for } \frac{1}{4} \le c \le 1. \end{cases}$$
(10)

$$PDF(c) = \begin{cases} -3 \ln c - 6\sqrt{1 - 4c} \tanh^{-1} \sqrt{1 - 4c}, & \text{for } 0 \le c \le \frac{1}{4} \\ 3 \ln 4, & \text{for } c = \frac{1}{4} \\ -3 \ln c + 2\sqrt{4c - 1} \left(-\pi + 3 \tan^{-1} \sqrt{4c - 1}\right) & \text{for } \frac{1}{4} \le c \le 1. \end{cases}$$
(11)

The graphs of these two distributions (10, 11) are displayed in Figure 4.



Figure 4. Plots of CDF(c) and PDF(c).

Future research directions

We would like to extend the current findings in several directions. First, the case of tetrahedron inscribed-tetrahedron picking appears as a natural extension. Next, the number theoretic properties of the integer sequence in (3) and the corresponding central moments will be investigated. Moreover, properties of inscribed triangles of area 1/4 appears as an interesting question. Finally, we notice that when we extend PDF(*c*) from (1/4, 1), as a complex function, to (0, 1/4) then its real part is same as the PDF(*c*) for (0, 1/4). An explanation of this phenomenon would be of interest.

Appendix

The Mathematica commands used to investigate the contours of Q and perform the resulting integration are summarized in an online appendix.

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Summary. Given a triangle, we derive the probability distribution function and the moments of the area of an inscribed triangle whose vertices are uniformly, and independently distributed on different sides of the given triangle. The theoretical results are confirmed by a Monte Carlo simulation and explored using a computer algebra system.

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